3.2 Operations which preserve convexity

2 vs ~~B~~ 13

Announcements

1. A1 due tomorrow
2. ~~Midterm~~?

① Non-negative weighted sums

If $f_1, \dots, f_m$ are convex (concave), weights $w_1, \dots, w_m \geq 0$, then

$$f = w_1 f_1 + \dots + w_m f_m \text{ is convex (resp. concave).}$$

Note: this means that the set of convex (concave) fcns is a convex cone!

② Compose w affine mapping

Consider $g(x) = f(Ax+b)$ for $f: \mathbb{R}^n \mapsto \mathbb{R}$, $A \in \mathbb{R}^{n \times m}$, $b \in \mathbb{R}^n$

$g: \mathbb{R}^m \mapsto \mathbb{R}$, $\mathrm{dom}(g) = \{x \mid Ax+b \in \mathrm{dom}(f)\}$

If f is convex (concave), then g is convex (resp. concave).

③ Pointwise maximum and supremum

max

If $f_1, \ldots, f_m$ are convex, then their pt-wise max f is convex, ie

$$f(x) = \max(f_1(x), \ldots, f_m(x)).$$

↳ pf via Jensen's inequality

sup

What if # of f_i is not finite?

Let's define functions $f(x,y)$ for $y \in A$. possibly infinite

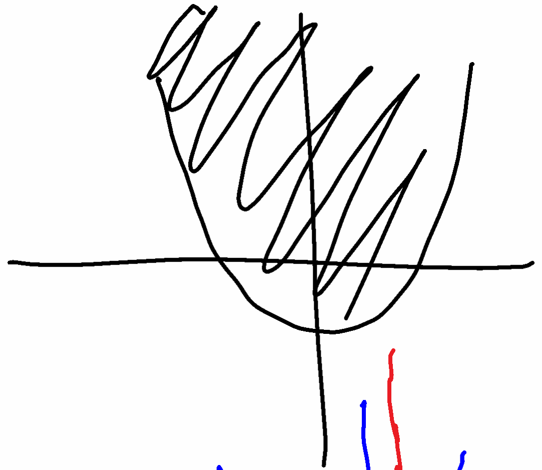
Consider $g(x) = \sup_{y \in A} f(x,y)$ $\quad \mathrm{dom}(g) = \{x \mid (x,y) \in \mathrm{dom}(f)\ \forall y \in A,\ \sup_{y \in A} f(x,y) < \infty\}$

If $\forall y \in A$, $f(x,y)$ is convex in x, $\quad (x, y \in \mathbb{R}^n,\ f: \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R})$

then g is convex.

(similarly: $\inf_{y \in A} f(x,y)$ is concave if $\forall y \in A$, $f(x,y)$ is concave).

Recall: $g(x) = \sup_{y\in A} f(x,y)$ Claim: If $f(x,y)$ is convex $\forall\, y \in A$, then g is convex on x.



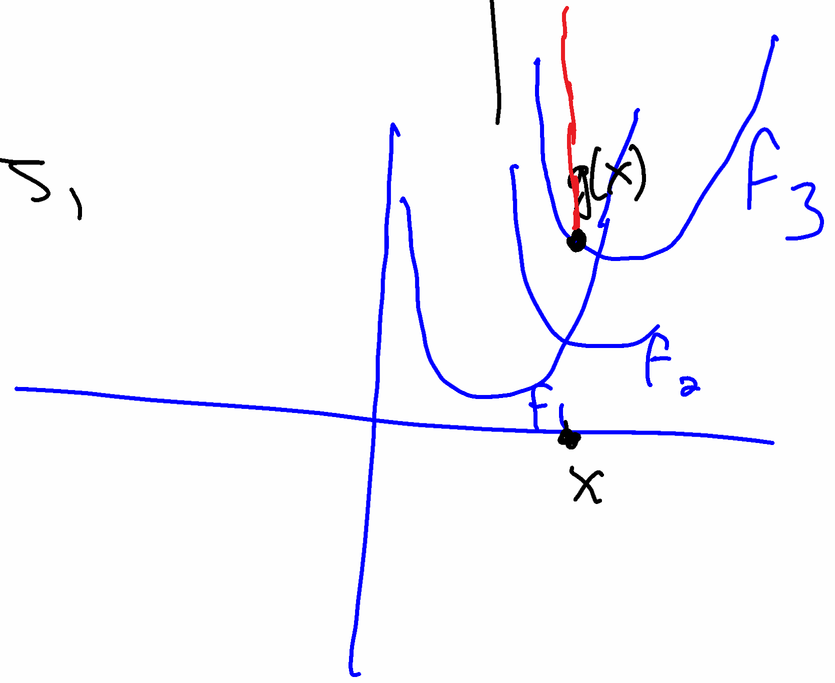
3 line proof using epigraphs:

Since $\forall\, y\in A$, $f(x,y)$ is convex, $\text{epi}(f(x,y))$ is convex set.

Moreover, $\text{epi}(g) = \bigcap_{y\in A} \text{epi}(f(\cdot,y))$

$\therefore$ $\text{epi}(g)$ is intersection of convex sets,

so $\text{epi}(g)$ is convex $\Rightarrow$ g convex. ∎



e.g. 3.7 Support fcn of a set

For $\emptyset \neq C \subseteq \mathbb{R}^n$, its support fcn S_C is

$$S_C(x) = \sup\{x^T y \mid y \in C\} \qquad \text{dom}(S_C) = \{x \mid \sup_{y\in C} x^T y < \infty\}$$

For any fixed $y \in \mathbb{R}^n$, the function $x^T y$ is affine $\Rightarrow$ convex

$\therefore$ $S_C(x)$ is pt-wise sup of family convex fcns, ie $S_C(x)$ is convex.

Ex 3.9 Least squares cost as a fcn of weights

For $a_1, \cdots, a_n \in \mathbb{R}^m$ and (possibly negative) weights $w_1, \cdots, w_m \in \mathbb{R}$, we want to consider:

fcn of weights!

$$g(w) = \inf_{x \in \mathbb{R}^m} \sum_{i=1}^{n} w_i (a_i^T x - b)^2$$

$$\mathrm{dom}(g) = \left\{ w \;\middle|\; \inf_{x} \sum_{i=1}^{n} w_i (a_i^T x - b) > -\infty \right\}$$

For any fixed x, $\sum_{i=1} w_i (a_i^T x - b)^2$ is a linear fcn in w.

$\therefore$ Since g is infimum of family of linear (hence concave) fcns,

g is concave in w.

Ex 3.10 Max eigenvalue of a symmetric matrix

Given $X \in S^m$, consider

$$f(X) = \lambda_{max}(X) = \sup\{y^T X y \mid \|y\|_2 = 1\}$$

largest eval

For any fixed y, $y^T X y$ is linear w.r.t. X.

$\therefore$ $f(X) = \lambda_{max}(X)$ is convex.

(4) Composition of functions (more generally)

Consider $f(x) = h(g(x))$, $h : \mathbb{R}^k \mapsto \mathbb{R}$, $g : \mathbb{R}^n \mapsto \mathbb{R}^k$, $f : \mathbb{R}^n \mapsto \mathbb{R}$

$\mathrm{dom}(f) = \{ x \in \mathrm{dom}(g) \mid g(x) \in \mathrm{dom}(h) \}$

Q: What conditions on h & g ensure f is convex?

Intuition: $n = 1$, $k = 1$

Assume h, g twice differentiable, $\mathrm{dom}(f) = \mathrm{dom}(g) = \mathbb{R}$.

When is $f(x) = h(g(x))$ convex?

↳ $f''(x) \geq 0 \quad \forall x \in \mathbb{R}$

Apply chain rule: $f''(x) = h''(g(x))\, g'(x)^2 + h'(g(x))\, g''(x)$.

h'' non-negative → h convex, h' non-neg. → h non-decreasing, g'' non-negative → g convex

$f(x) = h(g(x))$ $\quad$ h, g convex, $\quad$ h is non-decr.

↳ Let's apply this!

Consider $\quad f(x) = e^{g(x)}$ $\quad$ $(h(x) = e^x)$

$\therefore$ if g is convex, so is f.

Q: What about fcns s.t. domain $\neq \mathbb{R}$?

eg $h(x) = \log x$, s.t. $\operatorname{dom}(h) = \mathbb{R}_{++}$.

For convex h, using extended-value extension $\tilde{h}(x)$, we have conditions:

(w/o assuming differentiability of h & g or that $\operatorname{dom}(g) = \mathbb{R}^n$ and $\operatorname{dom}(h) = \mathbb{R}$):

a) f is convex if h is convex, $\tilde{h}$ is non-decreasing, g convex

b) f is " " , $\tilde{h}$ is non-increasing, g concave

c) f is concave if h is concave, $\tilde{h}$ non-decreasing, g concave

d) " " , $\tilde{h}$ non-increasing, g convex.

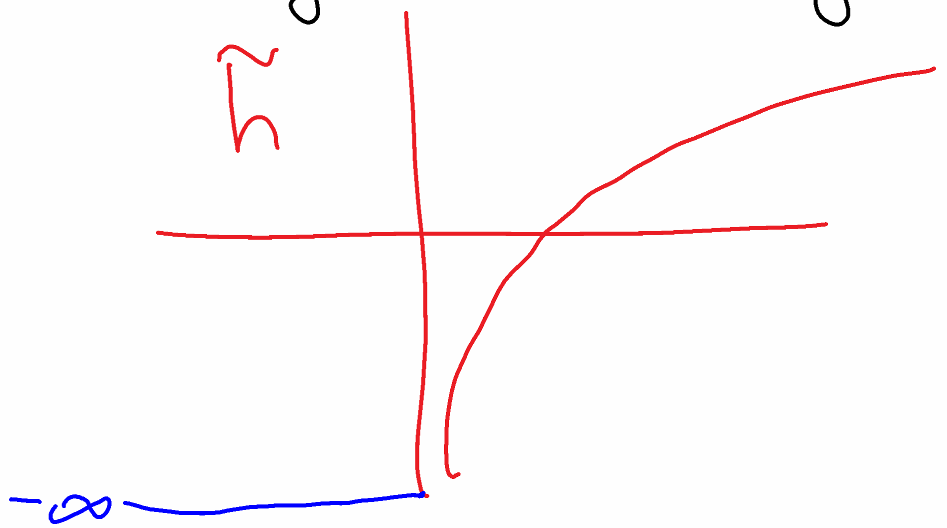
Note: - if h is convex $\tilde{h}(x) = \begin{cases} h(x) & \text{if } x \in \operatorname{dom} h \\ \infty & \text{o/w} \end{cases}$

- if h is concave $\tilde{h}(x) = \begin{cases} h(x) & \text{if } x \in \operatorname{dom} h \\ -\infty & \text{o/w} \end{cases}$

- a,b,c,d require $\tilde{h}$ non-incr (non-decr) on <u>all of $\mathbb{R}$</u>.

eg $h(x) = \log x$, $\operatorname{dom} h = \mathbb{R}_{++}$ is concave, $\qquad f(x) = h(g(x))$

$\tilde{h}$ non-decreasing on $\mathbb{R}$



$\therefore$ by (c), if g concave then $f(x) = \log(g(x))$ is concave.

Note: Similar statements if $k > 1$, i.e. $g: \mathbb{R}^n \mapsto \mathbb{R}^k$
$h: \mathbb{R}^k \mapsto \mathbb{R}$

See scn 3.2.4 of text for details.

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(5) Minimization

Consider $g(x) = \inf_{y \in C} f(x,y)$

If f is convex in (x,y), $C \neq \emptyset$ is a convex set, then g is convex.

$\mathrm{dom}(g) = \{x \mid (x,y) \in \mathrm{dom} f$ for some $y \in C\}$

e.g. 3.16 Distance to a set

Consider pt $x \in \mathbb{R}^n$ and set $S \subseteq \mathbb{R}^n$

$\mathrm{dist}(x, S) = \inf_{y \in S} \|x - y\|$ ← any norm

Since $\|x-y\|$ is convex in (x,y), if $S \neq \emptyset$ is convex, then $\mathrm{dist}(x, S)$ is convex in x.

e.g. 3.15 Consider $f(x,y) = x^T A x + 2x^T B y + y^T C y$ $\quad A, C \in S^n$

$$= [x^T \,\vdots\, y^T] \underbrace{\begin{bmatrix} A & B \\ B^T & C \end{bmatrix}}_{D} \begin{bmatrix} x \\ y \end{bmatrix}$$

(assign: $x^T A x + b^T x + c$ is convex iff $A \succeq 0$)

$f(x,y)$ is convex in (x,y) iff $D \succeq 0$.

$\therefore$ $g(x) = \inf_y f(x,y)$ is convex whenever $D \succeq 0$.

<u>Aside:</u> $g(x) = x^T (A - B C^{+} B^T) x$

C^{+}: pseudo-inverse of $C \in S^n$.